

Chaotic behavior of disordered nonlinear lattices

Haris Skokos

**Department of Mathematics and Applied Mathematics,
University of Cape Town
Cape Town, South Africa**

**E-mail: haris.skokos@uct.ac.za
URL: <http://www.mth.uct.ac.za/~hskokos/>**

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Outline

- **Disordered lattices:**
 - ✓ The quartic Klein-Gordon (KG) model
 - ✓ The disordered nonlinear Schrödinger equation (DNLS)
 - ✓ Different dynamical behaviors
- **Chaotic behavior of the KG model**
 - ✓ Lyapunov exponents
 - ✓ Deviation Vector Distributions
- **Numerical methods**
 - ✓ Symplectic Integrators
 - ✓ Tangent Map method
 - ✓ Different integration schemes for DNLS
- **Summary**

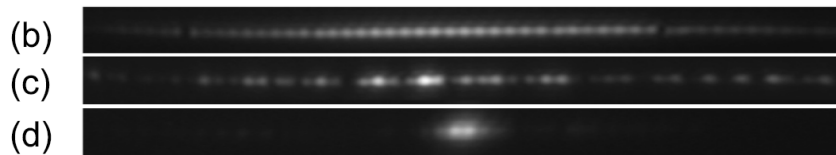
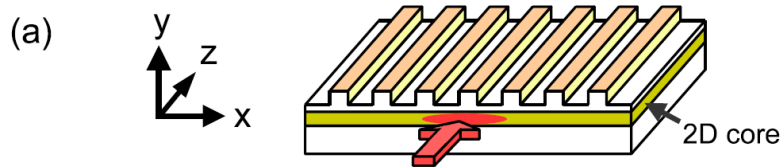
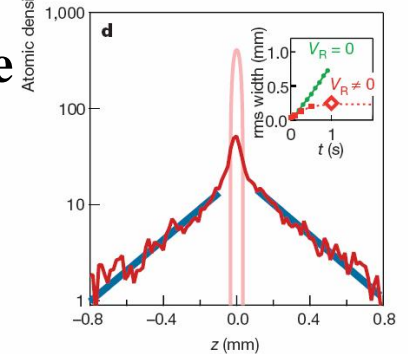
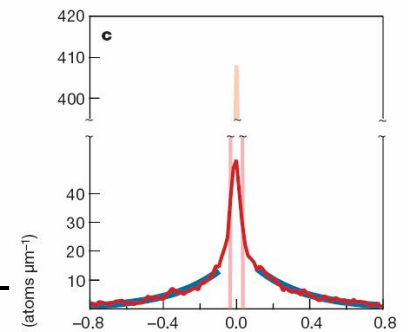
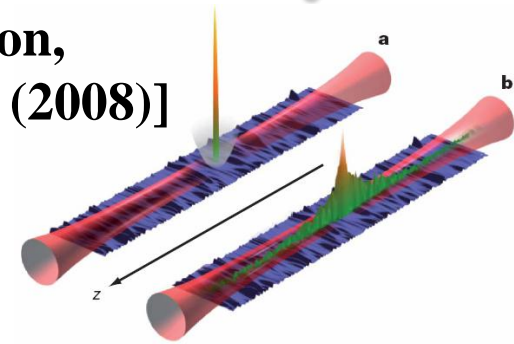
Interplay of disorder and nonlinearity

Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]

Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Lapyteva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]



The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^N \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} (u_{l+1} - u_l)^2$$

with **fixed boundary conditions** $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically $N=1000$.

Parameters: **W** and the **total energy E**. $\tilde{\varepsilon}_l$ **chosen uniformly from** $\left[\frac{1}{2}, \frac{3}{2}\right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l = A_l \exp(i\omega t)$. **Normal modes (NMs) $A_{v,l}$ - Eigenvalue problem:**

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^N \varepsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - (\psi_{l+1} \psi_l^* + \psi_{l+1}^* \psi_l)$$

where ε_l **chosen uniformly from** $\left[-\frac{W}{2}, \frac{W}{2}\right]$ and β **is the nonlinear parameter**.

Conserved quantities: The energy and the norm $S = \sum_l |\psi_l|^2$ of the wave packet.

Distribution characterization

We consider normalized **energy distributions** in normal mode (NM) space

$$z_v \equiv \frac{E_v}{\sum_m E_m} \quad \text{with} \quad E_v = \frac{1}{2} \left(\dot{A}_v^2 + \omega_v^2 A_v^2 \right), \quad \text{where } A_v \text{ is the amplitude}$$

of the v th NM (KG) or **norm distributions** (DNLS).

Second moment:
$$m_2 = \sum_{v=1}^N (v - \bar{v})^2 z_v \quad \text{with} \quad \bar{v} = \sum_{v=1}^N v z_v$$

Participation number:
$$P = \frac{1}{\sum_{v=1}^N z_v^2}$$

measures the number of stronger excited modes in z_v .

Single mode $P=1$. Equipartition of energy $P=N$.

Scales

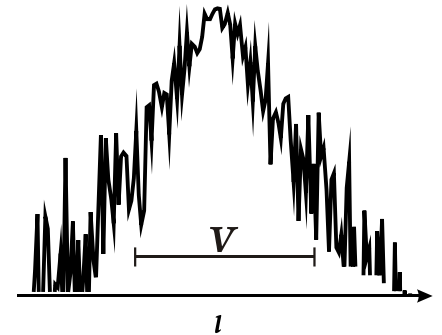
Linear case: $\omega_v^2 \in \left[\frac{1}{2}, \frac{3}{2} + \frac{4}{W} \right]$, width of the squared frequency spectrum:

$$\Delta_K = 1 + \frac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization
volume of an
eigenstate:

$$V \sim \frac{1}{\sum_{l=1}^N A_{v,l}^4}$$



Average spacing of squared eigenfrequencies of NMs within the range of a
localization volume: $d_K \approx \frac{\Delta_K}{V}$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{\epsilon}_l} \propto E \quad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \leq \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Different Dynamical Regimes

Three expected evolution regimes [Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Lapyteva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ : width of the frequency spectrum, d : average spacing of interacting modes, δ : nonlinear frequency shift.

Weak Chaos Regime: $\delta < d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d < \delta < \Delta$, $m_2 \sim t^{1/2} \rightarrow m_2 \sim t^{1/3}$

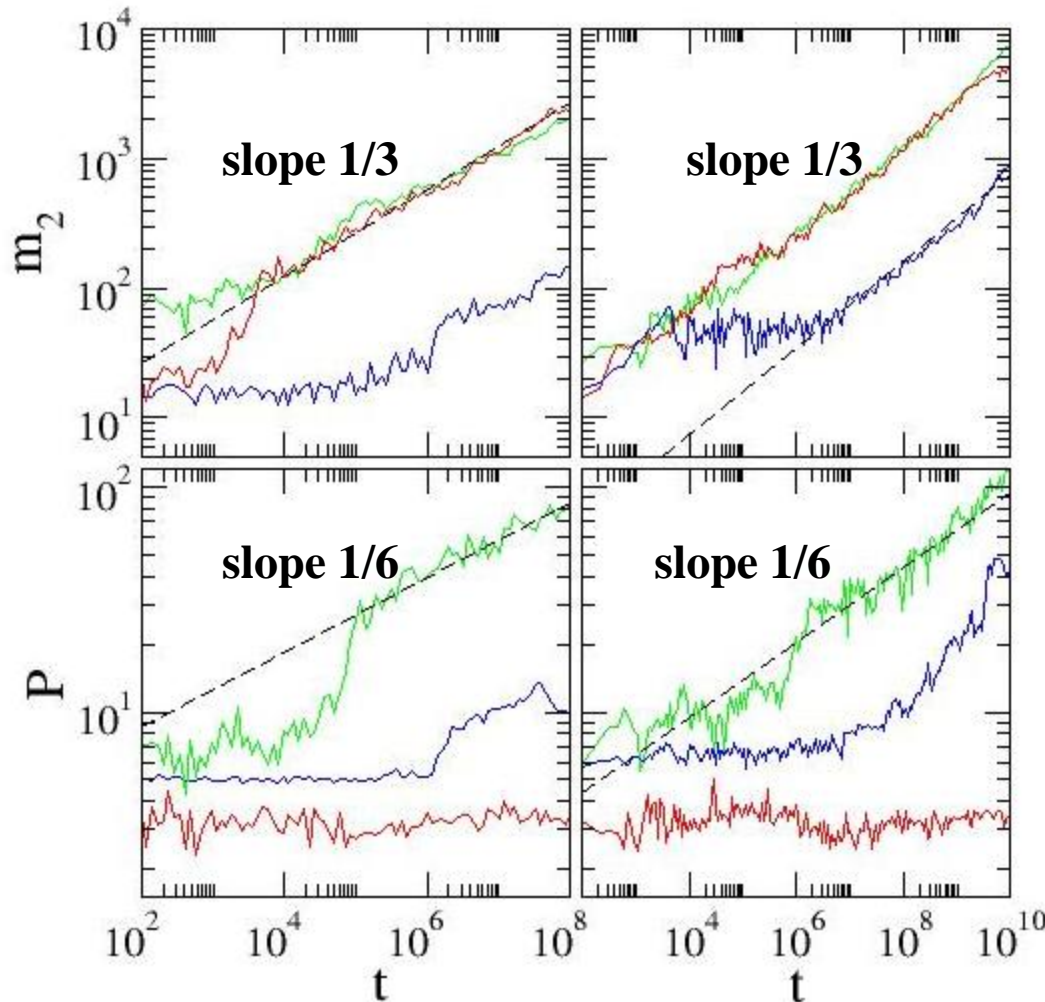
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta > \Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations

DNLS $W=4$, $\beta = 0.1, 1, 4.5$ **KG** $W = 4$, $E = 0.05, 0.4, 1.5$



No strong chaos regime

In weak chaos regime we averaged the measured exponent α ($m_2 \sim t^\alpha$) over 20 realizations:

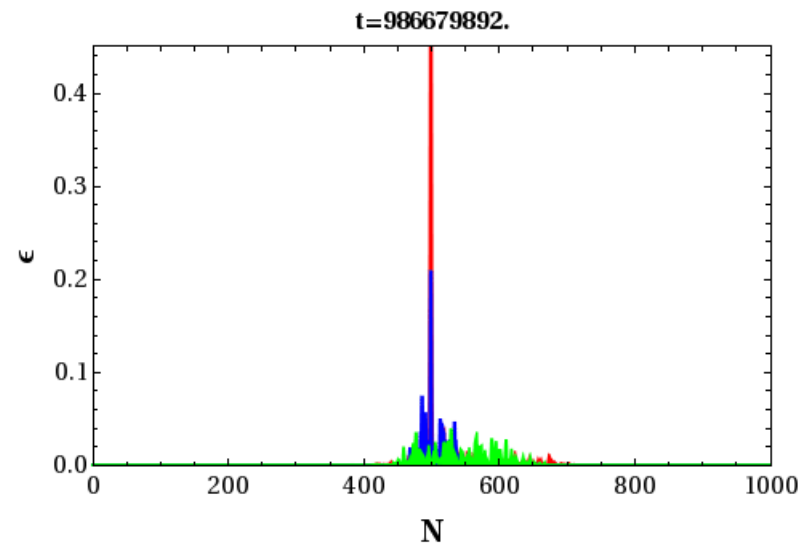
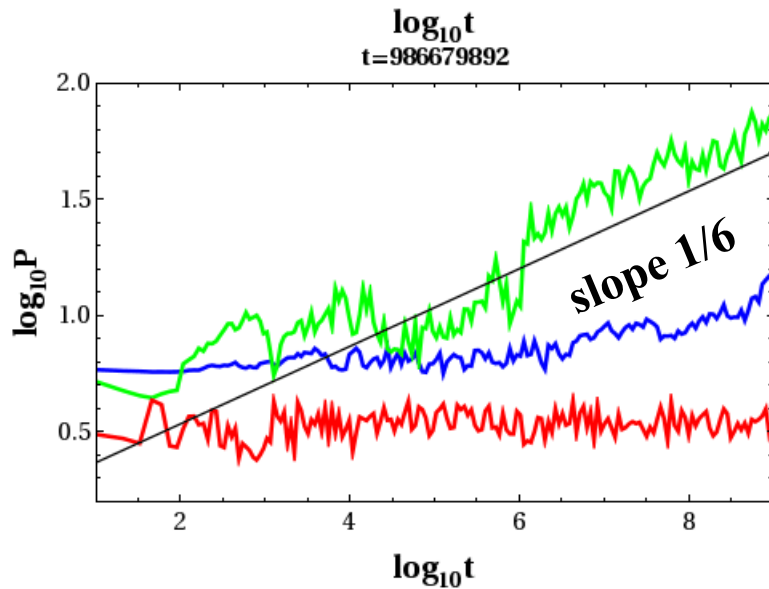
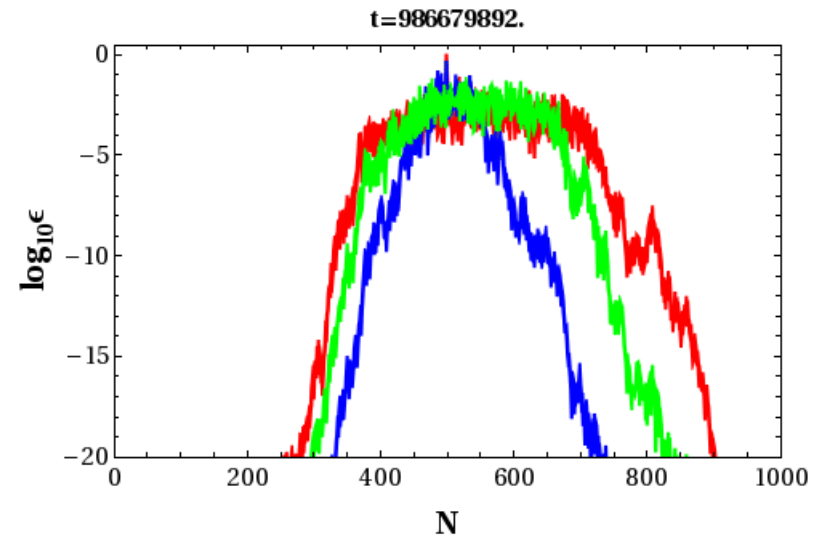
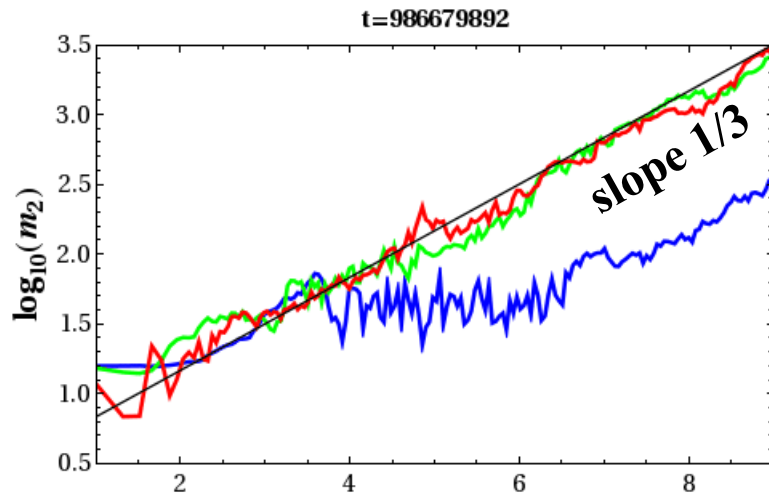
$$\alpha = 0.33 \pm 0.05 \text{ (KG)}$$

$$\alpha = 0.33 \pm 0.02 \text{ (DLNS)}$$

Flach et al., PRL (2009)

S. et al., PRE (2009)

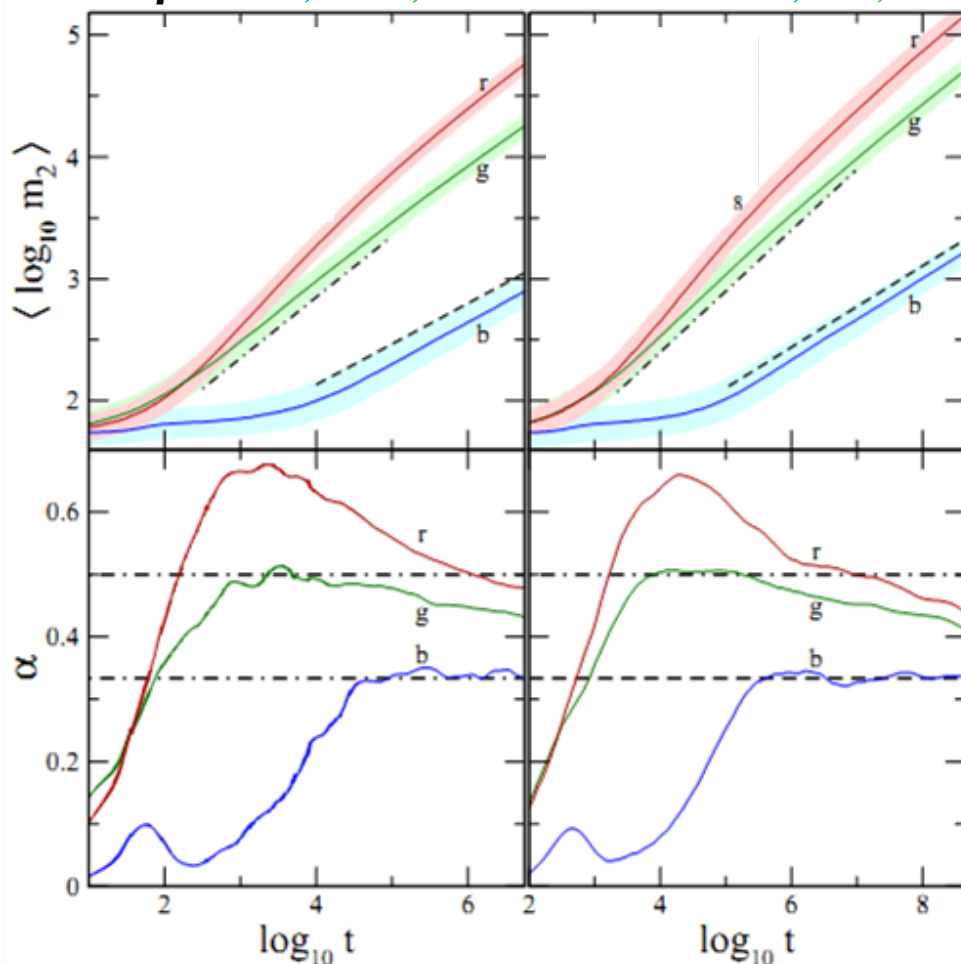
KG: Different spreading regimes



Crossover from strong to weak chaos (block excitations)

DNLS $\beta = 0.04, 0.72, 3.6$ KG $E = 0.01, 0.2, 0.75$

$W=4$



Average over 1000 realizations!

$$\alpha(\log t) = \frac{d \langle \log m_2 \rangle}{d \log t}$$

$\alpha=1/2$

$\alpha=1/3$

Laptyeva et al., EPL (2010)

Bodyfelt et al., PRE (2011)

Lyapunov Exponents (LEs)

Roughly speaking, the Lyapunov exponents of a given orbit characterize the **mean exponential rate of divergence** of trajectories surrounding it.

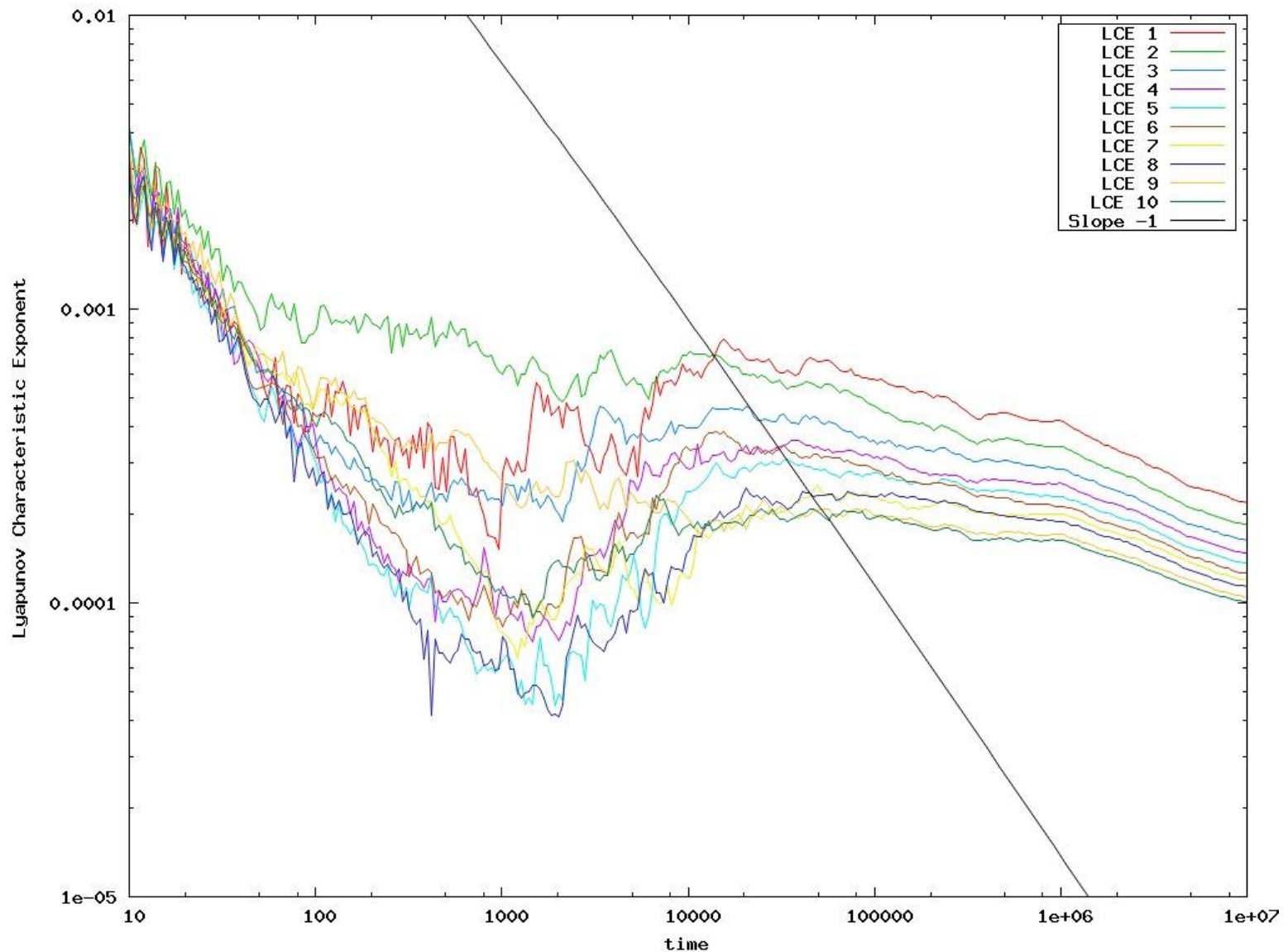
Consider an orbit in the $2N$ -dimensional phase space with **initial condition $\mathbf{x}(0)$** and an **initial deviation vector from it $\mathbf{v}(0)$** . Then the mean exponential rate of divergence is:

$$\text{mLCE} = \lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\vec{\mathbf{v}}(t)\|}{\|\vec{\mathbf{v}}(0)\|}$$

$\lambda_1 = 0 \rightarrow$ Regular motion $\propto (t^{-1})$

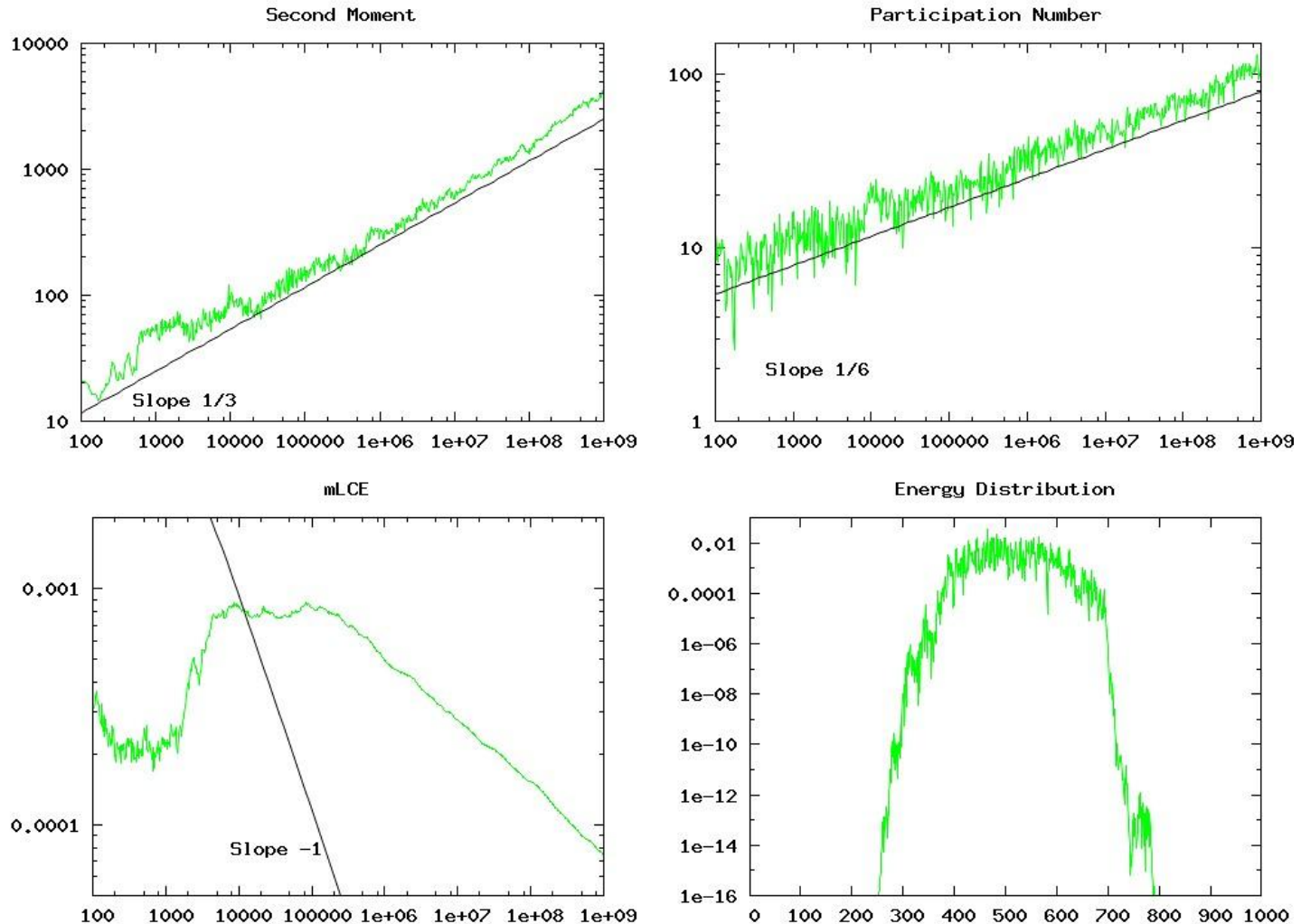
$\lambda_1 \neq 0 \rightarrow$ Chaotic motion

KG: LEs for single site excitations ($E=0.4$)



KG: Weak Chaos ($E=0.4$)

$t = 1000000000.00$

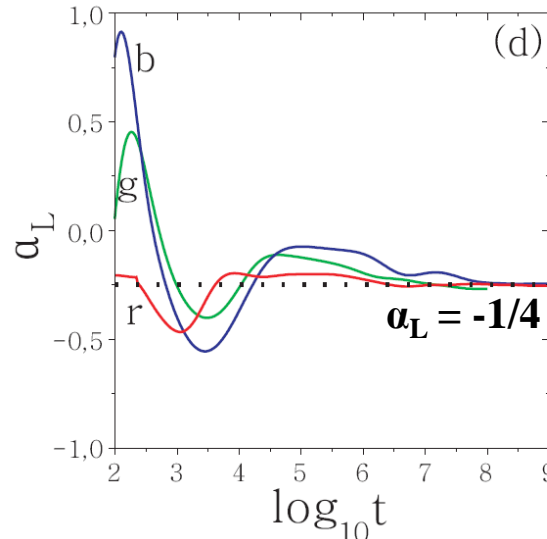
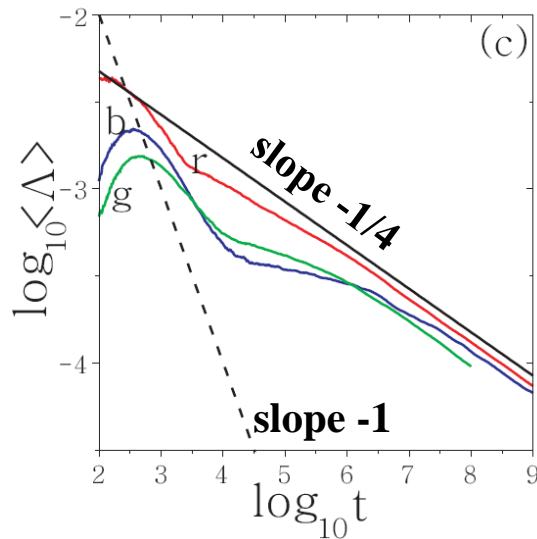
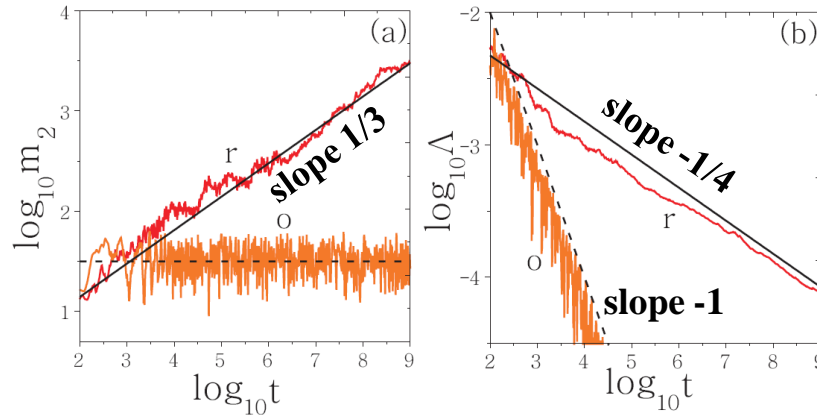


KG: Weak Chaos

Individual runs

Linear case

E=0.4, W=4



$$\alpha_L = \frac{d(\log \langle \Lambda \rangle)}{d \log t}$$

Average over 50 realizations

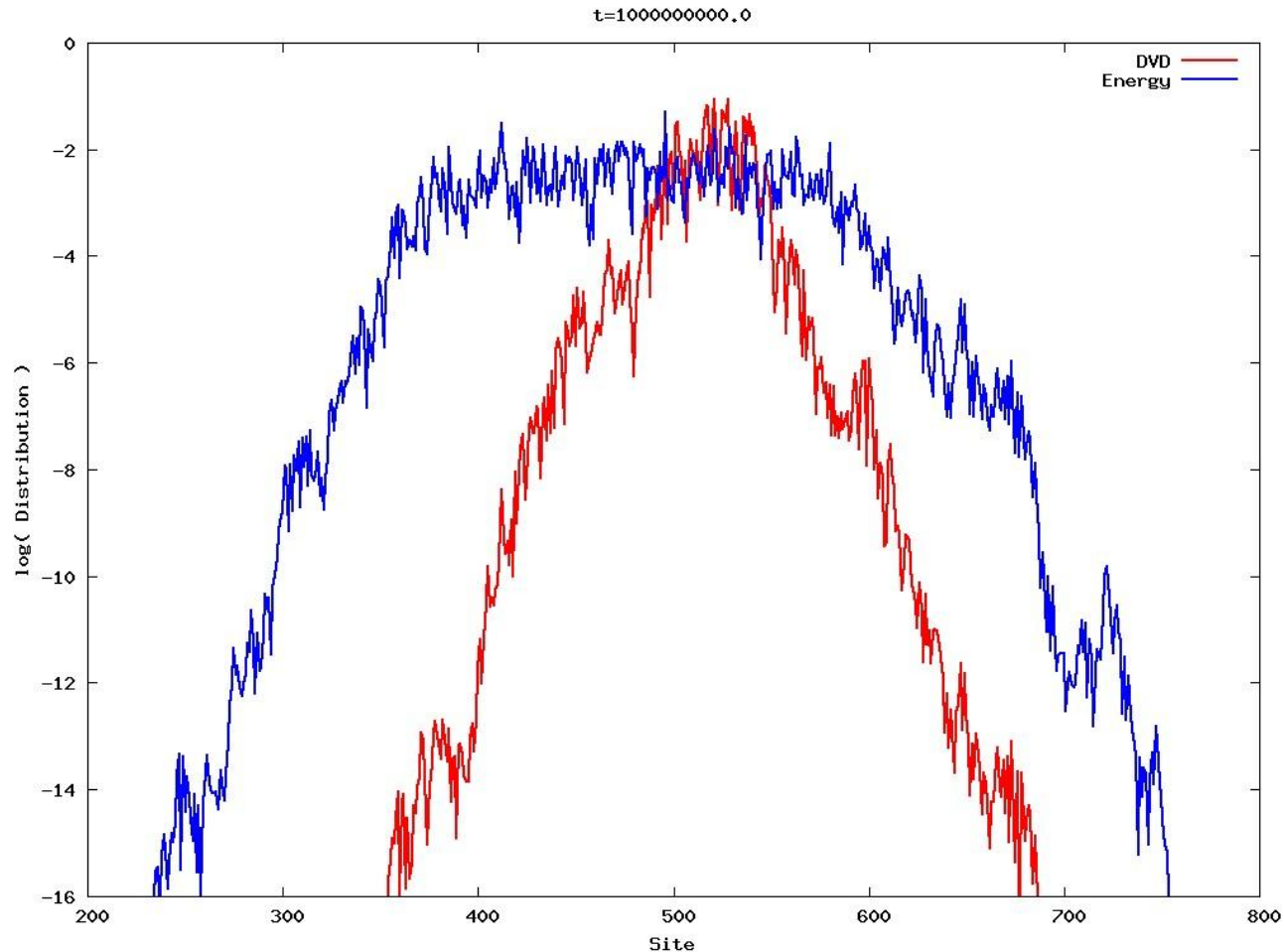
**Single site excitation E=0.4,
W=4**

**Block excitation (21 sites)
E=0.21, W=4**

**Block excitation (37 sites)
E=0.37, W=3**

S. et al. PRL (2013)

Deviation Vector Distributions (DVDs)

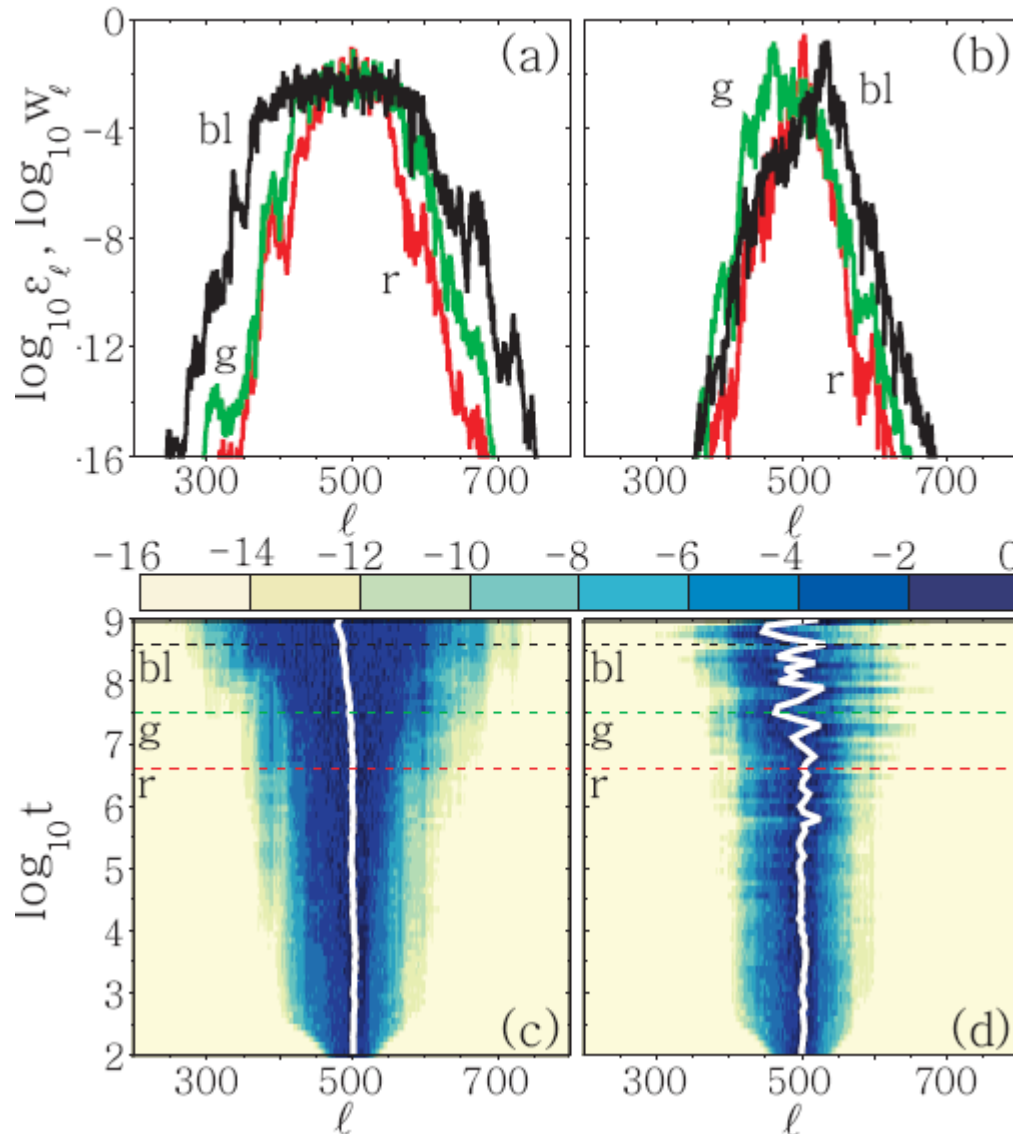


Deviation vector:

$$\mathbf{v}(t) = (\delta u_1(t), \delta u_2(t), \dots, \delta u_N(t), \delta p_1(t), \delta p_2(t), \dots, \delta p_N(t))$$

$$\text{DVD: } w_l = \frac{\delta u_l^2 + \delta p_l^2}{\sum_l (\delta u_l^2 + \delta p_l^2)}$$

Deviation Vector Distributions (DVDs)



Individual run
 $E=0.4, W=4$

Chaotic hot spots
meander through the
system, supporting a
homogeneity of chaos
inside the wave packet.

Integration scheme

Consider an **N degree of freedom** autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\overbrace{q_1, q_2, \dots, q_N}^{\text{positions}}, \overbrace{p_1, p_2, \dots, p_N}^{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0) = (q_1(0), q_2(0), \dots, q_N(0), p_1(0), p_2(0), \dots, p_N(0))$$

is governed by the **Hamilton's equations of motion**

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Autonomous Hamiltonian systems

Let us consider an **N degree of freedom** autonomous Hamiltonian systems of the form:

$$H(\vec{q}, \vec{p}) = \frac{1}{2} \sum_{i=1}^N p_i^2 + V(\vec{q})$$

As an example, we consider the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

Hamilton equations of motion:

$$\left\{ \begin{array}{l} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = y^2 - x^2 - y \end{array} \right.$$

Variational equations:

$$\left\{ \begin{array}{l} \dot{\delta x} = \delta p_x \\ \dot{\delta y} = \delta p_y \\ \dot{\delta p}_x = -(1 + 2y)\delta x - 2x\delta y \\ \dot{\delta p}_y = -2x\delta x + (-1 + 2y)\delta y \end{array} \right.$$

Symplectic Integrators (SIs)

Formally the solution of the Hamilton equations of motion can be written as:

$$\frac{d\vec{X}}{dt} = \{H, \vec{X}\} = L_H \vec{X} \Rightarrow \vec{X}(t) = \sum_{n \geq 0} \frac{t^n}{n!} L_H^n \vec{X} = e^{tL_H} \vec{X}$$

where \vec{X} is the full coordinate vector and L_H the Poisson operator:

$$L_H f = \sum_{j=1}^N \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}$$

If the Hamiltonian H can be **split into two integrable parts** as $H=A+B$, a symplectic scheme for integrating the equations of motion **from time t to time $t+\tau$** consists of approximating the operator $e^{\tau L_H}$ by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} = \prod_{i=1}^j e^{c_i \tau L_A} e^{d_i \tau L_B} + O(\tau^{n+1})$$

for appropriate values of constants c_i, d_i . This is **an integrator of order n** .

So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B .

Symplectic Integrator SABA₂C

The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)]:

$$SABA_2 = e^{c_1 \tau L_A} e^{d_1 \tau L_B} e^{c_2 \tau L_A} e^{d_1 \tau L_B} e^{c_1 \tau L_A}$$

with $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$, $c_2 = \frac{\sqrt{3}}{3}$, $d_1 = \frac{1}{2}$.

The integrator has only **small positive steps** and its **error is of order 2**.

In the case where **A is quadratic in the momenta and B depends only on the positions** the method can be improved by introducing a corrector C , having a small negative step:

$$C = e^{-\tau^3 \frac{c}{2} L_{\{\{A,B\}, B\}}}$$

with $c = \frac{2 - \sqrt{3}}{24}$.

Thus the full integrator scheme becomes: **$SABAC_2 = C (SABA_2) C$** and its **error is of order 4**.

Tangent Map (TM) Method

Any symplectic integration scheme used for solving the Hamilton equations of motion, which involves the act of Hamiltonians A and B, can be extended in order to integrate simultaneously the variational equations [S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].

The Hénon-Heiles system can be split as: $A = \frac{1}{2}(p_x^2 + p_y^2)$ $B = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$


$$\begin{array}{l}
 \dot{x} = p_x \\
 \dot{y} = p_y \\
 \dot{p}_x = -x - 2xy \\
 \dot{p}_y = y^2 - x^2 - y \\
 \delta\dot{x} = \delta p_x \\
 \delta\dot{y} = \delta p_y \\
 \delta\dot{p}_x = -(1 + 2y)\delta x - 2x\delta y \\
 \delta\dot{p}_y = -2x\delta x + (-1 + 2y)\delta y
 \end{array}
 \xrightarrow{A(\vec{p})}
 \left. \begin{array}{l}
 \dot{x} = p_x \\
 \dot{y} = p_y \\
 \dot{p}_x = 0 \\
 \dot{p}_y = 0 \\
 \delta\dot{x} = \delta p_x \\
 \delta\dot{y} = \delta p_y \\
 \delta\dot{p}_x = 0 \\
 \delta\dot{p}_y = 0
 \end{array} \right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u} \Rightarrow e^{\tau L_{AV}} : \left\{ \begin{array}{l}
 x' = x + p_x\tau \\
 y' = y + p_y\tau \\
 px' = p_x \\
 py' = p_y \\
 \delta x' = \delta x + \delta p_x\tau \\
 \delta y' = \delta y + \delta p_y\tau \\
 \delta p'_x = \delta p_x \\
 \delta p'_y = \delta p_y
 \end{array} \right.$$

$$\left. \begin{array}{l}
 \dot{x} = 0 \\
 \dot{y} = 0 \\
 \dot{p}_x = -x - 2xy \\
 \dot{p}_y = y^2 - x^2 - y \\
 \delta\dot{x} = 0 \\
 \delta\dot{y} = 0 \\
 \delta\dot{p}_x = -(1 + 2y)\delta x - 2x\delta y \\
 \delta\dot{p}_y = -2x\delta x + (-1 + 2y)\delta y
 \end{array} \right\} \xrightarrow{B(\vec{q})} \Rightarrow \frac{d\vec{u}}{dt} = L_{BV}\vec{u} \Rightarrow e^{\tau L_{BV}} : \left\{ \begin{array}{l}
 x' = x \\
 y' = y \\
 p'_x = p_x - x(1 + 2y)\tau \\
 p'_y = p_y + (y^2 - x^2 - y)\tau \\
 \delta x' = \delta x \\
 \delta y' = \delta y \\
 \delta p'_x = \delta p_x - [(1 + 2y)\delta x + 2x\delta y]\tau \\
 \delta p'_y = \delta p_y + [-2x\delta x + (-1 + 2y)\delta y]\tau
 \end{array} \right.$$


The KG model

We apply the **SABAC₂** integrator scheme to the KG Hamiltonian by using the **splitting**:

$$H_K = \sum_{l=1}^N \left(\underbrace{\frac{\mathbf{p}_l^2}{2}}_{\mathbf{A}} + \underbrace{\frac{\tilde{\epsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} (u_{l+1} - u_l)^2}_{\mathbf{B}} \right)$$



$$e^{\tau L_A}: \begin{cases} u'_l = p_l \tau + u_l \\ p'_l = p_l, \end{cases}$$



$$e^{\tau L_B}: \begin{cases} u'_l = u_l \\ p'_l = \left[-u_l(\tilde{\epsilon}_l + u_l^2) + \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right] \tau + p_l, \end{cases}$$

with a **corrector term** which corresponds to the Hamiltonian function:

$$\mathbf{C} = \{ \{ \mathbf{A}, \mathbf{B} \}, \mathbf{B} \} = \sum_{l=1}^N \left[u_l (\tilde{\epsilon}_l + u_l^2) - \frac{1}{W} (u_{l-1} + u_{l+1} - 2u_l) \right]^2.$$

The DNLS model

A **2nd order** SABA Symplectic Integrator with **5 steps**, combined with **approximate solution for the B part** (Fourier Transform): **SIFT²**

$$H_D = \sum_l \epsilon_l |\psi_l|^2 + \frac{\beta}{2} |\psi_l|^4 - (\psi_{l+1} \psi_l^* + \psi_{l+1}^* \psi_l), \quad \psi_l = \frac{1}{\sqrt{2}} (q_l + ip_l)$$

$$H_D = \sum_l \left(\underbrace{\frac{\epsilon_l}{2} (q_l^2 + p_l^2) + \frac{\beta}{8} (q_l^2 + p_l^2)^2}_{\mathbf{A}} - \underbrace{q_n q_{n+1} - p_n p_{n+1}}_{\mathbf{B}} \right)$$

$$e^{\tau L_A}: \begin{cases} q'_l = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p'_l = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \\ \alpha_l = \epsilon_l + \beta(q_l^2 + p_l^2)/2 \end{cases}$$

$$e^{\tau L_B}: \begin{cases} \varphi_q = \sum_{m=1}^N \psi_m e^{2\pi i q(m-1)/N} \\ \varphi'_q = \varphi_q e^{2i \cos(2\pi(q-1)/N) \tau} \\ \psi'_l = \frac{1}{N} \sum_{q=1}^N \varphi'_q e^{-2\pi i l(q-1)/N} \end{cases}$$

The DNLS model

Symplectic Integrators produced by **Successive Splits (SS)**

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2} (q_l^2 + p_l^2) + \frac{\beta}{8} (q_l^2 + p_l^2)^2}_{\text{A}} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{\text{B}} \right)$$

$$\left\{ \begin{array}{l} q'_l = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p'_l = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{array} \right. \left\{ \begin{array}{l} q'_l = q_l, \\ p'_l = p_l + (q_{l-1} + q_{l+1})\tau \end{array} \right. \left\{ \begin{array}{l} p'_l = p_l, \\ q'_l = q_l - (p_{l-1} + p_{l+1})\tau \end{array} \right.$$

Using the **SABA₂** integrator we get a **2nd order integrator with 13 steps, SS²:**

$$\text{SS}^2 = e^{\left[\frac{(3-\sqrt{3})}{6} \tau \right] L_A} \underbrace{e^{\frac{\tau}{2} L_B}}_{\text{A}} e^{\frac{\sqrt{3}\tau}{3} L_A} \underbrace{e^{\frac{\tau}{2} L_B}}_{\text{B}} e^{\left[\frac{(3-\sqrt{3})}{6} \tau \right] L_A}$$

$$\tau' = \tau / 2 \quad \underbrace{e^{\left[\frac{(3-\sqrt{3})}{6} \tau' \right] L_{B_1}} e^{\frac{\tau'}{2} L_{B_2}} e^{\frac{\sqrt{3}\tau'}{3} L_{B_1}} e^{\frac{\tau'}{2} L_{B_2}} e^{\left[\frac{(3-\sqrt{3})}{6} \tau' \right] L_{B_1}}}_{\text{A}} \underbrace{e^{\left[\frac{(3-\sqrt{3})}{6} \tau' \right] L_{B_1}} e^{\frac{\tau'}{2} L_{B_2}} e^{\frac{\sqrt{3}\tau'}{3} L_{B_1}} e^{\frac{\tau'}{2} L_{B_2}} e^{\left[\frac{(3-\sqrt{3})}{6} \tau' \right] L_{B_1}}}_{\text{B}}$$

Non-symplectic methods for the DNLS model

In our study we also use the **DOP853 integrator** which is an explicit non-symplectic Runge-Kutta integration scheme of order 8.

DOP853: Hairer et al. 1993,
<http://www.unige.ch/~hairer/software.html>

Three part split symplectic integrators for the DNLS model

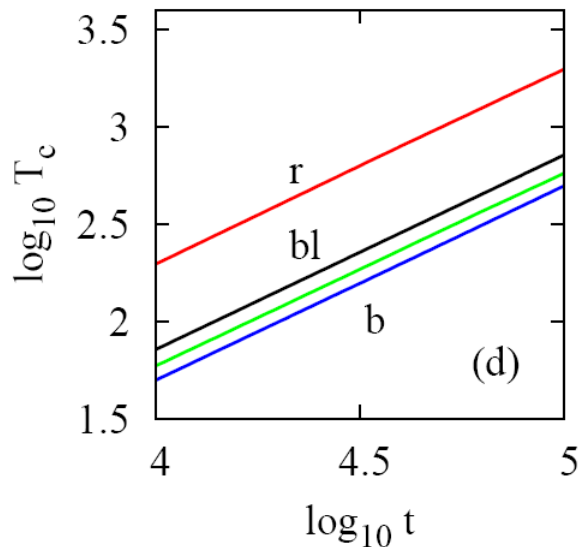
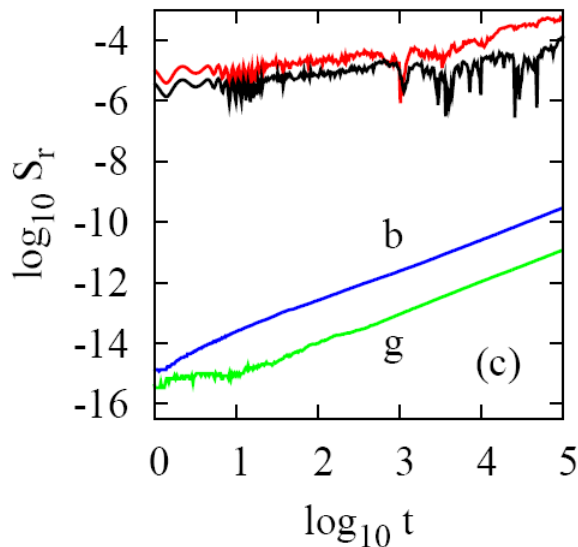
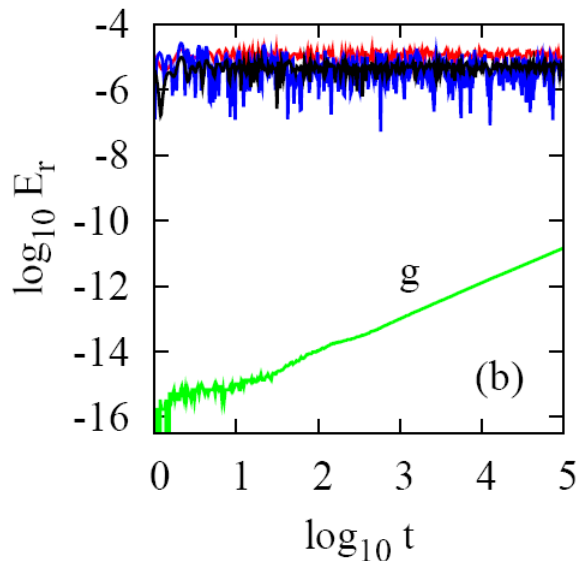
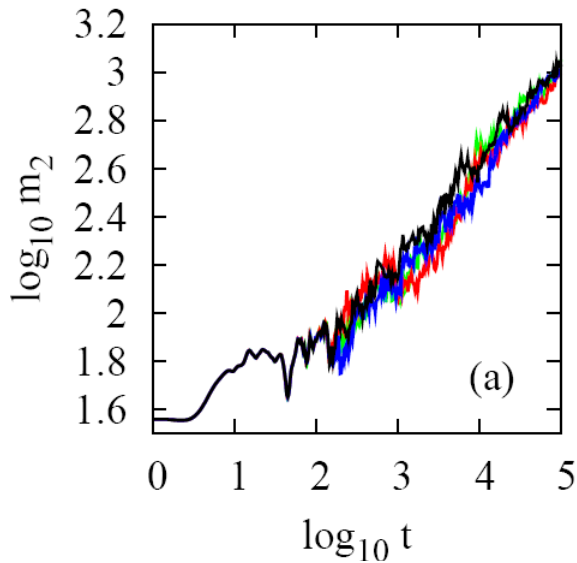
Three part split symplectic integrator of order 2, with 5
steps: ABC^2

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2} (q_l^2 + p_l^2) + \frac{\beta}{8} (q_l^2 + p_l^2)^2}_A \underbrace{-q_n q_{n+1}}_B \underbrace{-p_n p_{n+1}}_C \right)$$

$$ABC^2 = e^{\frac{\tau}{2} L_A} e^{\frac{\tau}{2} L_B} e^{\tau L_C} e^{\frac{\tau}{2} L_B} e^{\frac{\tau}{2} L_A}$$

This low order integrator has already been used by e.g. Chambers, MNRAS (1999) – Goździewski et al., MNRAS (2008).

2nd order integrators: Numerical results



ABC² $\tau=0.005$

SS² $\tau=0.02$

DOP853 $\delta=10^{-16}$

SIFT² $\tau=0.05$

E_r : relative energy error

S_r : relative norm error

T_c : CPU time (sec)

S. et al., Phys. Lett. A (2014)

Composition Methods: 4th order SIs

Starting from any 2nd order symplectic integrator S^{2nd} , we can construct a 4th order integrator S^{4th} using the **composition method** proposed by Yoshida [Phys. Lett. A (1990)]:

$$S^{4th}(\tau) = S^{2nd}(x_1\tau) \times S^{2nd}(x_0\tau) \times S^{2nd}(x_1\tau), \quad x_0 = -\frac{2^{1/3}}{2 - 2^{1/3}}, \quad x_1 = \frac{1}{2 - 2^{1/3}}$$

In this way, starting with the 2nd order integrators **SS²**, **SIFT²** and **ABC²** we construct the 4th order integrators:

SS⁴ with 37 steps

SIFT⁴ with 13 steps

ABC⁴_[Y] with 13 steps

Composition method proposed by Suzuki [Phys. Lett. A (1990)]:

$$S^{4th}(\tau) = S^{2nd}(p_2\tau) \times S^{2nd}(p_2\tau) \times S^{2nd}((1 - 4p_2)\tau) \times S^{2nd}(p_2\tau) \times S^{2nd}(p_2\tau)$$
$$p_2 = \frac{1}{4 - 4^{1/3}}, \quad 1 - 4p_2 = -\frac{4^{1/3}}{4 - 4^{1/3}}$$

Starting with the 2nd order integrators **ABC²** we construct the 4th order integrator: **ABC⁴_[S] with 21 steps.**

More 4th order SIs

We construct few more integration schemes by considering the 4th order symplectic integrators **ABA864**, **ABA1064**, **ABAH864** and **ABAH1064** introduced by Blanes et al., Appl. Num. Math. (2013) and Farrés et al., Cel. Mech. Dyn. Astr. (2013).

Approximating the solution of the B part by a Fourier Transform we construct the 4th order integrators:

SIFT⁴₈₆₄ with 43 steps

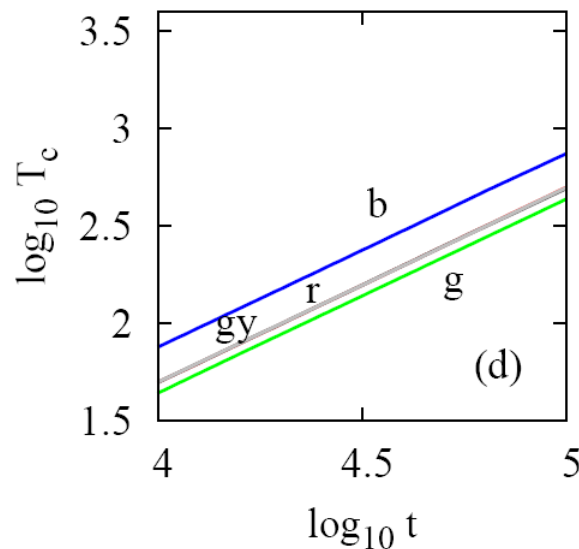
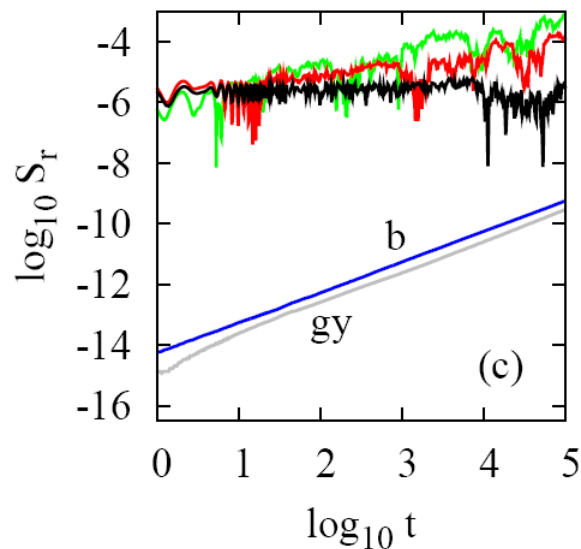
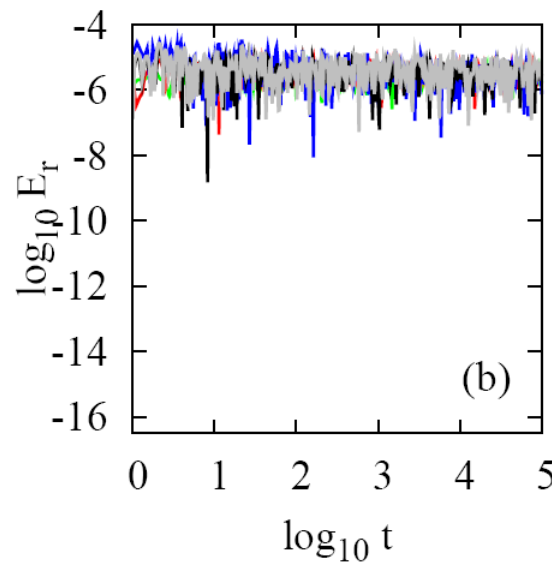
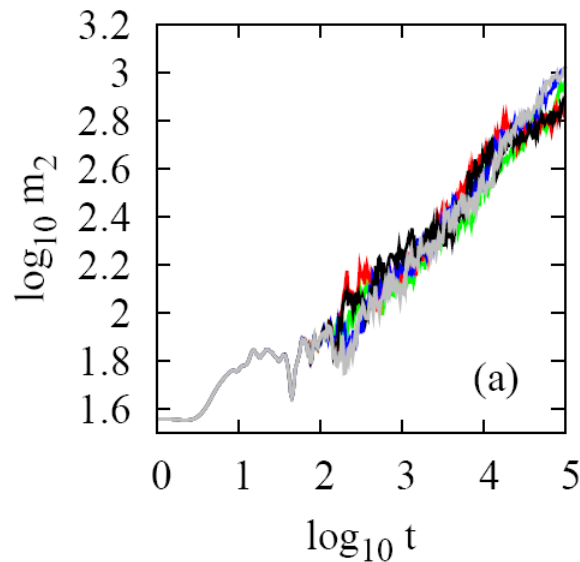
SIFT⁴₁₀₆₄ with 49 steps

Using **successive splits for the B part** and implementing the SABA₂ integrator for its integration, we construct the 4th order integrators:

SS⁴₈₆₄ with 49 steps

SS⁴₁₀₆₄ with 55 steps

4th order integrators: Numerical results (I)



SIFT⁴ $\tau=0.125$

SIFT² $\tau=0.05$

ABC⁴_[S] $\tau=0.1$

SS⁴ $\tau=0.1$

ABC⁴_[Y] $\tau=0.05$

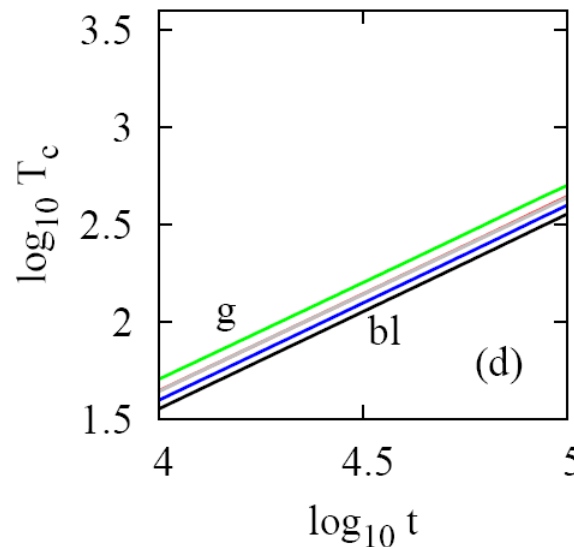
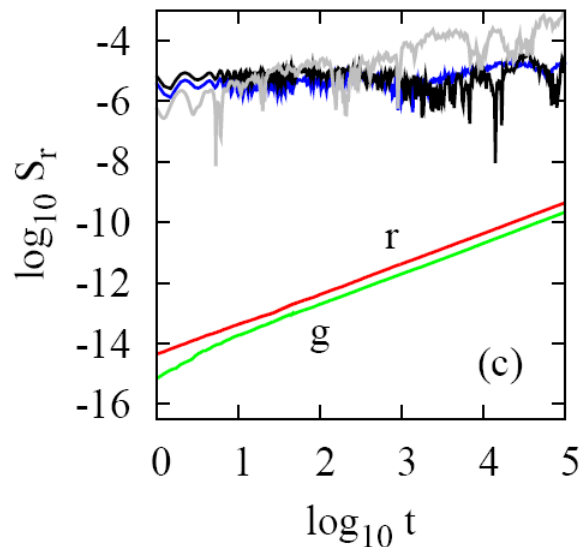
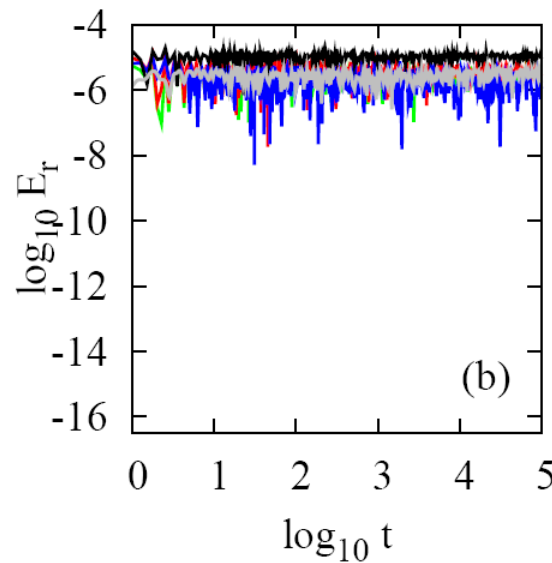
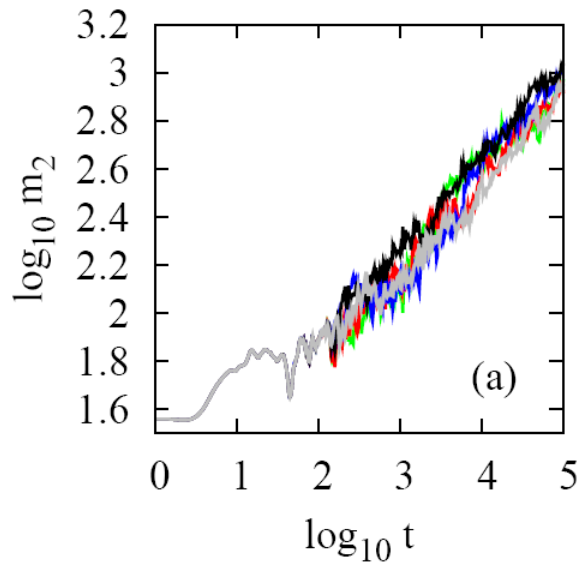
E_r : relative energy error

S_r : relative norm error

T_c : CPU time (sec)

S. et al., Phys. Lett. A (2014)

4th order integrators: Numerical results (II)



SIFT⁴₁₀₆₄ $\tau=0.25$

ABC⁴_[Y] $\tau=0.05$

SIFT⁴₈₆₄ $\tau=0.25$

SS⁴₁₀₆₄ $\tau=0.25$

SS⁴₈₆₄ $\tau=0.25$

E_r : relative energy error

S_r : relative norm error

T_c : CPU time (sec)

S. et al., Phys. Lett. A (2014)

High order composition methods (I)

Using a composition technique introduced by Yoshida [Phys. Lett. A (1990)] we construct the 6th order symplectic integrator $ABC^6_{[Y]}$ having 29 steps :

$$ABC^6(\tau) = ABC^2(w_3\tau) \times ABC^2(w_2\tau) \times ABC^2(w_1\tau) \times \\ \times ABC^2(w_0\tau) \times ABC^2(w_1\tau) \times ABC^2(w_2\tau) \times ABC^2(w_3\tau)$$

whose coefficients

$$w_1 = -1.17767998417887$$
$$w_2 = 0.235573213359357$$
$$w_3 = 0.784513610477560$$
$$w_0 = 1 - 2(w_1 + w_2 + w_3)$$

cannot be given in analytic form.

High order composition methods (II)

In addition, following the works of
Kahan & Li, Math Comput. (1997), and
Sofroniou & Spaletta, Optim. Methods Softw. (2005)
we implement some efficient **high order composition methods**,
considering as the basic block the 2nd order ABC² integrator.

ABC⁶_[KL] with 37 steps

ABC⁶_[SS] with 45 steps

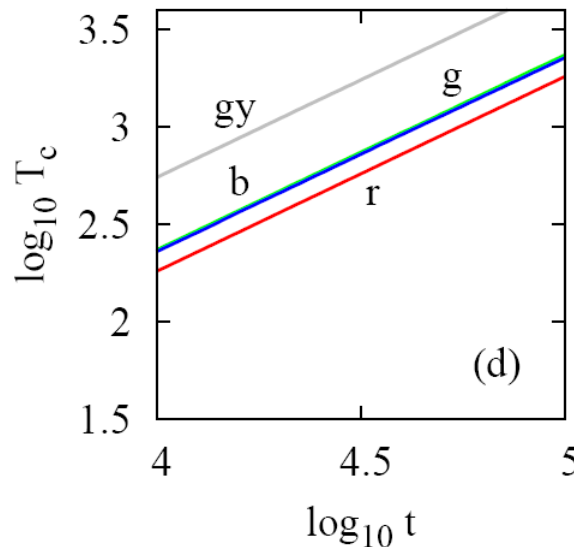
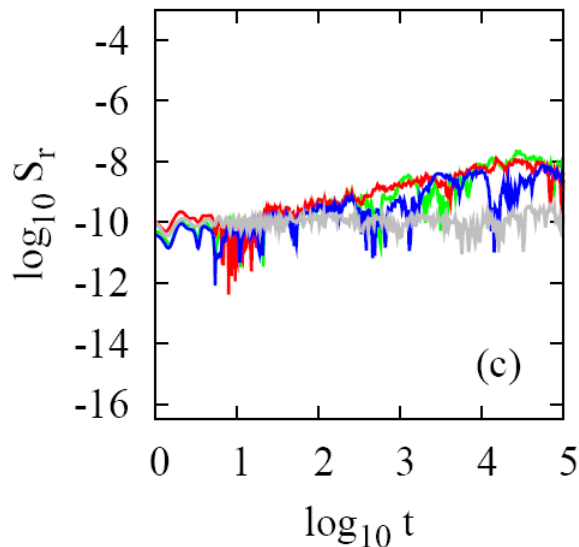
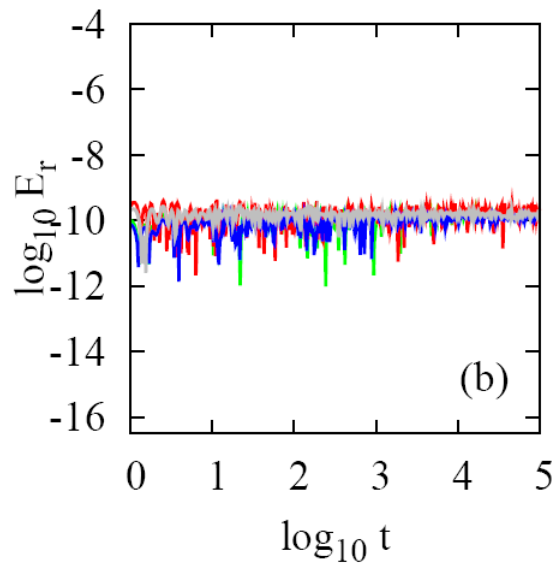
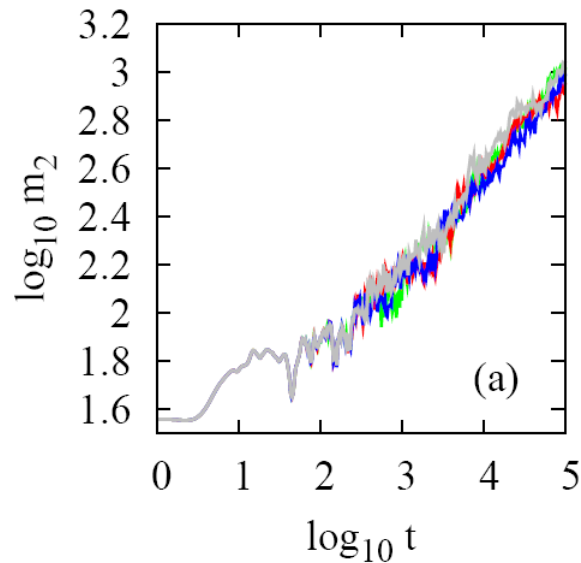
ABC⁸_[Y] with 61 steps

ABC⁸_[KL] with 69 steps

ABC⁸_[SS] with 77 steps

ABC¹⁰_[SS] with 125 steps

High order integrators: Numerical results (I)



$SS^4_{864} \tau=0.015625$

$ABC^6_{[Y]} \tau=0.03$

$ABC^6_{[KL]} \tau=0.04$

$ABC^6_{[SS]} \tau=0.125$

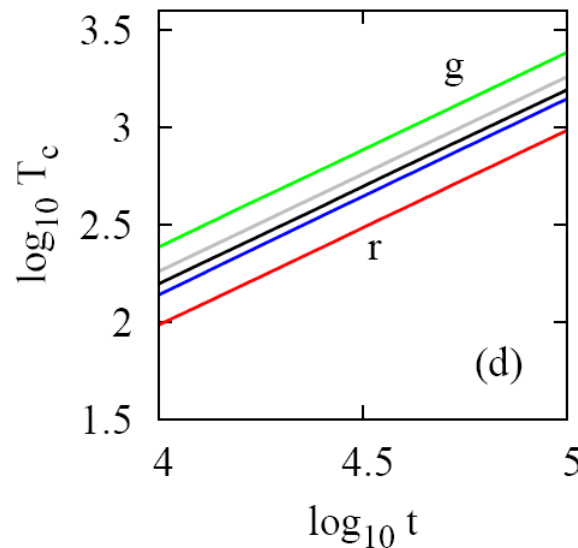
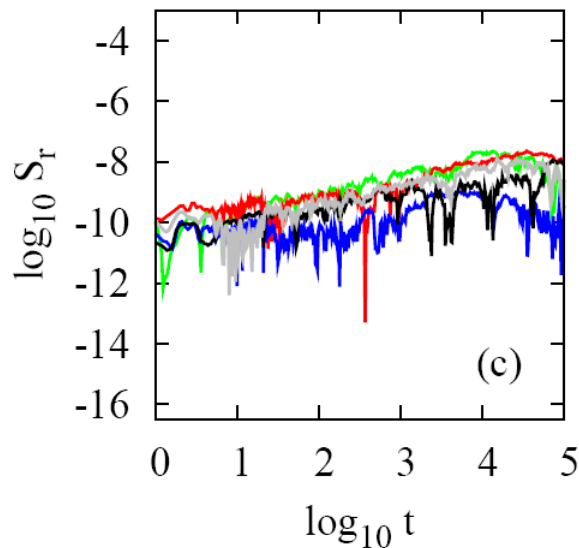
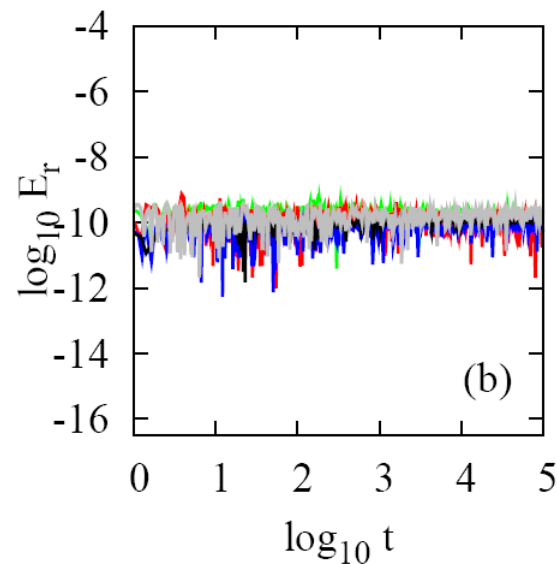
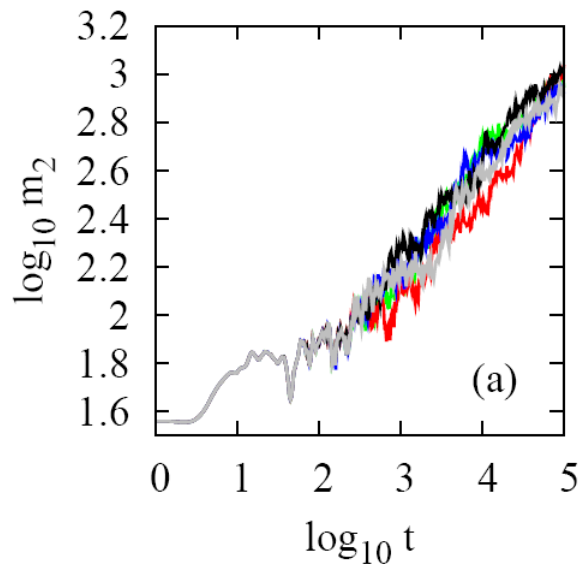
E_r : relative energy error

S_r : relative norm error

T_c : CPU time (sec)

S. et al., Phys. Lett. A (2014)

High order integrators: Numerical results (II)



ABC⁸_[Y] $\tau=0.0625$

ABC⁶_[SS] $\tau=0.125$

ABC¹⁰_[SS] $\tau=0.2$

ABC⁸_[KL] $\tau=0.125$

ABC⁸_[SS] $\tau=0.2$

E_r : relative energy error

S_r : relative norm error

T_c : CPU time (sec)

S. et al., Phys. Lett. A (2014)

Summary (I)

- We presented **three different dynamical behaviors** for wave packet spreading in 1d nonlinear disordered lattices:
 - ✓ **Weak Chaos Regime:** $\delta < d$, $m_2 \sim t^{1/3}$
 - ✓ **Intermediate Strong Chaos Regime:** $d < \delta < \Delta$, $m_2 \sim t^{1/2} \rightarrow m_2 \sim t^{1/3}$
 - ✓ **Selftrapping Regime:** $\delta > \Delta$
- **Generality of results:**
 - ✓ Two different models: KD and DNLS,
 - ✓ Predictions made for DNLS are verified for both models.
- **Lyapunov exponent** computations show that:
 - ✓ Chaos not only exists, but also persists.
 - ✓ Slowing down of chaos does not cross over to regular dynamics.
 - ✓ Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.
- Our results suggest that **Anderson localization is eventually destroyed by nonlinearity**, since spreading does not show any sign of slowing down.

Summary (II)

- We presented several **efficient integration methods** suitable for the **integration of the DNLS model**, which are based on **symplectic integration techniques**.
- The construction of symplectic schemes based on **3 part split of the Hamiltonian** was emphasized (**ABC methods**).
- Algorithms based on the integration of the B part of Hamiltonian via **Fourier transforms**, i.e. methods SIFT^2 , SIFT^4 , SIFT^4_{864} and SIFT^4_{1064} succeeded in keeping the relative norm error S_r very low.
Drawback: they require the number of lattice sites to be 2^k , $k \in \mathbb{N}^*$.
- We hope that our results will **initiate future research** both for the theoretical development of new, improved 3 part split integrators, as well as for their applications to different dynamical systems.

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Thank you for your attention